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Multi-instanton and string loop corrections in toroidal orbifold models

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Abstract: We analyze $\mathcal{N}=2$ (perturbative and non-perturbative) corrections to the effective theory in type I orbifold models where a dual heterotic description is available. These corrections may play an important role in phenomenological scenarios. More precisely, we consider two particular compactifications: the Bianchi-Sagnotti-Gimon-Polchinski orbifold and a freely-acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with $\mathcal{N}=1$ supersymmetry and gauge group $\mathrm{SO}(q) \times \mathrm{SO}(32-q)$. By exploiting perturbative calculations of the physical gauge couplings on the heterotic side, we obtain multi-instanton and one-loop string corrections to the Kähler potential and the gauge kinetic function for these models. The non-perturbative corrections appear as sums over relevant Hecke operators, whereas the one-loop correction to the Kähler potential matches the expression proposed in [1, 2]. We argue that these corrections are universal in a given class of models where target-space modular invariance (or a subgroup of it) holds.

Keywords: Superstrings and Heterotic Strings, String Duality, Nonperturbative Effects, Supersymmetric Effective Theories.

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Contents

1. Introduction

In the last years there has been a remarkable progress in understanding the structure of String Theory at tree-level in the perturbative expansion, that is, in the supergravity limit. Flux compactifications [3] have provided us with a helpful framework which partially addresses long-standing problems such as moduli stabilization or supersymmetry breaking. However, despite this progress, the resulting message continues to be that non-perturbative and string loop corrections play an indispensable role, mainly due to the generic presence of remnant flat directions in the scalar potential and the difficulties of obtaining a chiral spectrum in their absence.

The computation of α' and non-perturbative corrections to the effective theory is still in an early stage, even for cases where a description in terms of a free CFT is available. Much effort has been pursued on understanding the role of $\mathcal{N}=1$ euclidean brane instantons [4]. These may turn out to be useful for generating new couplings in the superpotential [5–9], moduli-stabilization [10] or supersymmetry breaking [11]. Moreover, the one-loop string corrections to the Kähler potential computed in [12], have been shown to play an important role in large volume scenarios [13], leading to a hierarchy of mass scales without the necessity of a big amount of fine-tuning. Additional one-loop string corrections to the Kähler potential have been computed in [1, 2] for toroidal compactifications (see also [14, 15]).

In this note, we analyze multi-instanton and one-loop string corrections arising from $\mathcal{N}=2$ sectors in toroidal orbifold compactifications of type I String Theory. These generically correct the Kähler potential, K, and the gauge kinetic function, f_a, of the effective theory, and therefore may play an important role in phenomenological scenarios with classical flat directions. For this aim, we follow the techniques introduced in [16], and build type I-heterotic S-dual pairs of orbifold models.

Schematically, the procedure can be summarized as follows. The one-loop physical gauge couplings in the heterotic side take the expression,

$$4\pi^2 g_a^{-2}(\mu^2)|_{1-\text{loop}} = \frac{k_a}{\ell} + \frac{b_a}{4}\log\frac{M_\text{s}^2}{\mu^2} + \frac{\Delta_a(M,\bar{M})}{4}\,, \tag{1.1}$$

with ℓ the linear multiplet associated to the dilaton, M_s the string scale, M the moduli of the compactification and k_a the normalization of the gauge group generators, determined by the level of the corresponding Kac-Moody algebra. The β-function coefficient, b_a, is given in terms of the quadratic Casimir invariants of the gauge group,

$$b_a = \sum_r n_r T_a(r) - 3T_a(\text{adj}_a)\,, \tag{1.2}$$

with n_r the number of matter multiplets in the representation r.

On the other hand, the field theory result reads [17, 18],

$$\begin{aligned} 4\pi^2 g_a^{-2}(\mu^2)|_{1-\text{loop}} = \text{Re } f_a(M)|_{1-\text{loop}} + \frac{b_a}{4}\left(\log\frac{M_\text{Planck}^2}{\mu^2} - \log(S+\bar{S})\right) \\ +\frac{1}{4}\left(c_a\hat{K}(M,\bar{M}) - 2\sum_r T_a(r)\log\det Z_r(M,\bar{M})\right)\,, \end{aligned} \tag{1.3}$$

where det Z_r is the determinant of the tree-level Kähler metric associated to the matter multiplets in the representation r, $\hat{K}(M,\bar{M})$ the tree-level Kähler potential for the moduli M and,

$$c_a = \sum_r n_r T_a(r) - T(\text{adj}_a)\,. \tag{1.4}$$

In order to compare (1.1) and (1.3), it is convenient to express the relation between the usual complex axiodilaton S and the linear multiplet ℓ as,

$$\frac{1}{\ell} = \text{Re } S - \frac{1}{4}\Delta_\text{univ.} \tag{1.5}$$

with $\Delta_{\text{univ.}}$ a gauge group independent ("universal") function. In what follows we split $\Delta_{\text{univ.}}$ in its harmonic and non-harmonic parts,

$$\Delta_{\text{univ.}}(M,\bar{M}) = V_{(1)}(M,\bar{M}) + H(M) + H^*(\bar{M}) \ . \tag{1.6}$$

In terms of these, the Kähler potential and the gauge-kinetic function of the $\mathcal{N}=1$ effective theory are given to one-loop by [18 – 20],[1]

$$K|_{1-\text{loop}} = -\log\left(S+\bar{S}-\frac{1}{2}V_{(1)}(M,\bar{M})\right) + \hat{K}(M,\bar{M})\,, \tag{1.7}$$

$$\begin{aligned}\text{Re } f_a|_{1-\text{loop}} = k_a\text{Re } S + \frac{1}{4}\Big(&\Delta_a(M,\bar{M}) - V_{(1)}(M,\bar{M}) - c_a\hat{K}(M,\bar{M}) \\ &- 2\sum_r T_a(r)\log\det Z_r(M,\bar{M})\Big)\end{aligned} \tag{1.8}$$

We can then reinterpret the results in terms of $E1$ multi-instanton and string loop corrections on the type I side, as K and f_a should be invariant (up to Kähler transformations) under S-duality transformations.

In the present work, we consider in detail two classes of models on which the heterotic S-dual partition function can be easily worked out: the Bianchi-Sagnotti-Gimon-Polchinsky (BSGP) orbifold [21, 22], with gauge group $\mathrm{U}(16)\times\mathrm{U}(16)$, and the $\mathbb{Z}_2\times\mathbb{Z}_2$ freely-acting orbifold model with gauge group $\mathrm{SO}(q)\times\mathrm{SO}(32-q)$, presented in [16] and based on the model of [23] (see also [24]). The motivation is multiple. First, based on the modular symmetries preserved by the scalars in Calabi-Yau compactifications, and more precisely on the axionic shift symmetries, it has been argued in [25] that non-perturbative corrections to the gauge kinetic function should appear in an exponentiated form, $\exp(2\pi M)$, with M the corresponding moduli. Our results on the BSGP orbifold show indeed that global symmetries constrain very much the shape of multi-instanton and string-loop corrections and, with few extra assumptions, they are completely determined in models on which modular invariance in the moduli space of the compactification applies.[2]

The freely-acting $\mathrm{SO}(q)\times\mathrm{SO}(32-q)$ orbifold model represents, on the other hand, an example on which part of the original modular symmetry is broken by the compactification. A remarkable fact, already pointed out in [16], is that $E1$ instantons in this model always appear within multiplets under the orbifold action (doublets or quadruplets). We believe that this may be a general feature of flux compactifications, where the fluxes gauge some of the original symmetries and induce non-trivial discrete torsion. It is therefore interesting to see how non-perturbative and loop corrections are affected by the "background" in this simple example. Moreover, it was also observed in [16] that, in the heterotic S-duals, the orbifold action on the winding modes was different for $q=0 \pmod 8$ or $q=4 \pmod 8$,

[1] We have defined S in such a way that the harmonic part of the universal threshold, $H(M)$, naturally corrects the holomorphic gauge kinetic function. Notice that this is always possible since the chiral field S has no fixed relation to vertices of string theory.

[2] For previous works on the role of global symmetries for determining non-perturbative and string loop corrections, see [26, 28].

pointing out a possible dependence of the type I instantonic effects on the rank of the gauge group. It is also our aim to make this dependence explicit.

Although similar computations to the ones performed here have been carried out e.g. for R^4 [26, 27], F^4 [27, 29 − 31] and the four hyperini [32] couplings, this is to our knowledge the first explicit computation of stringy multi-instanton corrections to the Kähler potential and the gauge kinetic function. We hope that these results will help to shed some light on some of the issues raised in the previous paragraphs, and more interestingly, to clarify the possible role of these corrections in phenomenological scenarios.

The paper is organized as follows. In section 2 we construct the partition function for the heterotic S-dual of the BSGP orbifold model, and extract the $E1$ multi-instanton and one-loop string corrections to the Kähler potential and the gauge kinetic function in the resulting effective theory. In section 3 we proceed similarly with the $\mathrm{SO}(q) \times \mathrm{SO}(32-q)$ freely acting orbifold model. We comment on the possible "universality" of some of our results and discuss possible generalizations in section 4. Finally, we give some concluding remarks in section 5. We have relegated all the details on the computations to the appendix, in order not to overload the bulk of the paper with many technicalities.

2. The Bianchi-Sagnotti-Gimon-Polchinski orbifold

In this section we consider the BSGP type I orbifold model [21, 22], corresponding to the $T^4/\mathbb{Z}_2 \times T^2$ orbifold limit of type I String Theory compactified on $K3 \times T^2$. In order to cancel the RR tadpoles, 8 D5-branes and 16 D9-branes are required. For D5 branes lying on top of an orbifold fixed point, the complete massless spectrum has a $\mathrm{U}(16) \times \mathrm{U}(16)$ gauge group with hypermultiplets in $2(\mathbf{120}, \mathbf{1}) + 2(\mathbf{1}, \mathbf{120}) + (\mathbf{16}, \mathbf{16})$. In the Coulomb branch, where a half D5-brane is located at each of the 16 fixed points, the Green-Schwarz mechanism takes place and only the $\mathrm{U}(16)$ gauge group from the D9-branes remains massless, with spectrum given by four hypermultiplets, containing the moduli of the $K3$, three vector multiplets containing the axiodilaton and the moduli of the T^2, a $\mathbf{120} + \overline{\mathbf{120}}$, and sixteen $\mathbf{16}$ coming from the D5-D9 modes. The coefficient of the β-function turns out to be $b_{\mathrm{U}(16)} = 12$. Perturbative threshold corrections to gauge couplings [38] depend on the moduli of T^2, denoted T_1 and U_1 in what follows. Since the dilaton S and T_1, U_1 are in $\mathcal{N} = 2$ vector multiplets in 4d language, this is consistent with supersymmetry. A priori we expect non-perturbative corrections to depend nontrivially on the three vector multiplets, and to be insensitive to the $T^4/\mathbb{Z}_2$ moduli, called $T_{2,3}$ and $U_{2,3}$ in what follows. We will show, by performing explicitly the computation using the heterotic S-dual, that this expectation is indeed correct.[3]

2.1 Heterotic S-dual partition function

We want to find the one-loop partition function for the heterotic dual of the BSGP model, proposed in [39] for the above Coulomb branch. In [40] it was shown that this corresponds

[3] Due to the nature of our approach, based on type I-heterotic duality, all the computations are carried out at the orbifold point. In particular, we do not consider blow-up modes which would lead us out of the orbifold point, and are massive due to the above Green-Schwarz mechanism.

to a standard SO(32) heterotic $T^2 \times T^4/\mathbb{Z}_2$ orbifold with shift vector $V = \frac{1}{4}(1,\dots,1,-3)$. The various orbifold blocks are then as follows. The left-moving fermions contribute as,[4]

$$Z_{\rm L}\begin{bmatrix} h \\ g \end{bmatrix} = \frac{1}{2\eta^4}\sum_{a,b=0}^{1}(-1)^{a+b+ab+bh}\vartheta^2\begin{bmatrix} a \\ b \end{bmatrix}\vartheta^2\begin{bmatrix} a+h \\ b+g \end{bmatrix} , \tag{2.1}$$

with $h, g = 0, 1$ labelling the different untwisted and twisted orbifold sectors.
Analogously, the bosonic T^4 blocks read,

$$Z_{(4,4)}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{\hat{Z}_2\hat{Z}_3}{|\eta|^8} \quad \text{and} \quad Z_{(4,4)}\begin{bmatrix} h \\ g \end{bmatrix} = \left|\frac{2\eta}{\vartheta\begin{bmatrix} 1-h \\ 1-g \end{bmatrix}}\right|^4 , \quad \text{for } hg \neq 0 \ . \tag{2.2}$$

Here we have defined the toroidal lattice sums $\hat{Z}_r$ as,

$$\hat{Z}_r = \frac{\text{Re } T_r}{\tau_2}\sum_{n_1,\ell_1,n_2,\ell_2}\exp\left[-2\pi T_r\det(A) - \frac{\pi(\text{Re } T_r)}{\tau_2(\text{Re } U_r)}\left|\begin{pmatrix} 1 & iU_r \end{pmatrix} A \begin{pmatrix} \tau \\ -1 \end{pmatrix}\right|^2\right] , \tag{2.3}$$

$$A = \begin{pmatrix} n_1 & \ell_1 \\ n_2 & \ell_2 \end{pmatrix} , \tag{2.4}$$

with n_i and ℓ_i integers. For the right moving fermions we find,

$$\Gamma\begin{bmatrix} h \\ g \end{bmatrix} = \frac{1}{2\bar{\eta}^{16}}\sum_{a,b=0}^{1}(-1)^{ga+hb}e^{-\frac{i\pi hg}{2}}\bar{\vartheta}^{16}\begin{bmatrix} a-\frac{h}{2} \\ b-\frac{g}{2} \end{bmatrix} . \tag{2.5}$$

Putting everything together we finally get the one-loop partition function for the heterotic dual of the BSGP model,

$$\begin{aligned} T = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^3}\frac{\hat{Z}_1}{4|\eta|^8}\Bigg[&(Q_o+Q_v)\frac{\hat{Z}_2\hat{Z}_3}{|\eta|^8}\Gamma\begin{bmatrix} 0 \\ 0 \end{bmatrix} + (Q_o-Q_v)\left|\frac{2\eta}{\vartheta_2}\right|^4\Gamma\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &+(Q_s+Q_c)\left|\frac{2\eta}{\vartheta_4}\right|^4\Gamma\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (Q_s-Q_c)\left|\frac{2\eta}{\vartheta_3}\right|^4\Gamma\begin{bmatrix} 1 \\ 1 \end{bmatrix}\Bigg] , \end{aligned} \tag{2.6}$$

where,

$$Q_o + Q_v = \frac{1}{2\eta^4}(\vartheta_3^4 - \vartheta_1^4 - \vartheta_2^4 - \vartheta_4^4) , \tag{2.7}$$

$$Q_o - Q_v = \frac{1}{2\eta^4}(\vartheta_3^2\vartheta_4^2 - \vartheta_4^2\vartheta_3^2 - \vartheta_1^2\vartheta_2^2 - \vartheta_2^2\vartheta_1^2) , \tag{2.8}$$

$$Q_s - Q_c = \frac{1}{2\eta^4}(\vartheta_1^2\vartheta_3^2 + \vartheta_3^2\vartheta_1^2 + \vartheta_4^2\vartheta_2^2 - \vartheta_2^2\vartheta_4^2) , \tag{2.9}$$

$$Q_s + Q_c = \frac{1}{2\eta^4}(\vartheta_3^2\vartheta_2^2 - \vartheta_2^2\vartheta_3^2 + \vartheta_4^2\vartheta_1^2 + \vartheta_1^2\vartheta_4^2) . \tag{2.10}$$

[4]In order not to overload the expressions with notation, we have omitted the arguments of the modular functions. Unless explicitly stated, these are implicitly understood to be functions of τ. Moreover, theta functions are evaluated at the point $\nu = 0$, as usual. For definitions of the various modular functions, affine characters and orbifold blocks, see e.g. [41, 42].

2.2 Perturbative and non-perturbative corrections

Following the general discussion around (1.1), our task here is to compute the one-loop threshold corrections to the physical gauge coupling in the heterotic model (2.6), as these are mapped to one-loop and $E1$ multi-instanton corrections in the BSGP orbifold. In terms of the partition function, these are given by [33–35],

$$\Lambda \equiv \frac{b_{\mathrm{U}(16)}}{4}\log\frac{M_s^2}{\mu^2}+\frac{\Delta_{\mathrm{U}(16)}}{4}=\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\frac{i}{4\pi}\frac{1}{|\eta|^4}\sum_{a,b=0}^{1}\partial_\tau\left(\frac{\vartheta\left[{a \atop b}\right]}{\eta}\right)\left(Q^2-\frac{1}{4\pi\tau_2}\right)C\left[{a \atop b}\right], \quad (2.11)$$

where Q is the charge operator of the corresponding gauge group, and $C\left[{a \atop b}\right]$ is the internal six-dimensional partition function. Following the same procedure as in [16] we find,

$$\Lambda = -\frac{1}{8}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\hat{Z}_1\hat{\mathcal{A}}_f \quad (2.12)$$

with,[5]

$$\hat{\mathcal{A}}_f = -\frac{1}{20\eta^{24}}(D_{10}E_{10}-48\eta^{24}) = \frac{1}{12\eta^{24}}\left(\hat{E}_2E_4E_6-\frac{5}{12}E_6^2-\frac{7}{12}E_4^3\right), \quad (2.13)$$

and $\hat{Z}_1$ given in (2.3). The definitions of the Eisenstein series, E_{2k}, can be found for instance in the appendix of [16].

The details of the computation are in appendix A.1. Notice that the numerator of $\hat{\mathcal{A}}_f$ is an almost-holomorphic modular form, its non-holomorphicity being exclusively due to the presence of $\hat{E}_2$,

$$\hat{E}_2 \equiv E_2 - \frac{3}{\pi\tau_2}\ . \quad (2.14)$$

As it will be made more explicit below, these non-holomorphic terms can be traced back to perturbative and non-perturbative corrections to the Kähler potential of the effective theory.

Both, $\hat{Z}_1$ and $\hat{\mathcal{A}}_f$, are invariant under the full modular group Γ, so we can directly apply the method of Dixon-Kaplunovsky-Louis (DKL) [34] to evaluate the integral in (2.12). This consists of depicting the lattice sum, $\hat{Z}_1$, into orbits under the modular group, and evaluate the integral for each class of orbits in a suitable unfolded region of the upper complex half-plane. The matrices (2.4) can be classified in three kind of orbits under the modular group,

1. *Zero orbit:*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

[5] The modular covariant derivative D_d is defined as,

$$D_d = \frac{i}{\pi}\partial_\tau + \frac{d/2}{\pi\tau_2}$$

2. *Non-degenerate orbits:*
$$\begin{pmatrix} k & j \\ 0 & p \end{pmatrix} ,$$
with $k > j \geq 0$, $p \neq 0$ and $AV = AV'$ iff $V = V'$, for $V, V' \in \Gamma$.

3. *Degenerate orbits:*
$$\begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix} ,$$
with $(j, p) \sim (-j, -p)$ and $AV = AV'$ iff $V = T^n V'$, for some integer n and $V, V' \in \Gamma$.

We therefore unfold (2.12) into three integrals corresponding to the above representatives. Non-degenerate orbits are integrated over the double cover of the upper half complex plane, $\mathbb{C}^+$, whereas degenerate orbits have to be integrated over the fundamental domain, $\mathcal{F}_T$, of the subgroup generated by T, for arbitrary j and p. The details of the computation can be found in appendix A.1. Putting all pieces together and disregarding constant terms arising from the regularization scheme, we obtain,

$$\begin{aligned} \Lambda = & \frac{\pi}{2} \text{Re } T_1 \; - \; 3 \left(\log|\eta(iU_1)|^4 + \log[(\text{Re } U_1)(\text{Re } T_1)\mu^2] \right) \; - \; \frac{\pi}{3} \frac{E(iU_1, 2)}{T_1 + \bar{T}_1} \\ & - \frac{1}{4} \left(\sum_{k>j\geq 0, p>0} \frac{1}{kp} e^{-2\pi pkT_1} \left[\hat{\mathcal{A}}_f(\mathcal{U}) + \frac{1}{\pi kp} \frac{\hat{\mathcal{A}}_K(\mathcal{U})}{T_1 + \bar{T}_1} \right] \; + \; \text{c.c.} \right) \end{aligned} \tag{2.15}$$

where $E(U, k)$ is the non-holomorphic Eisenstein series of order k, defined as

$$E(U, k) \equiv \frac{1}{\zeta(2k)} \sum_{(j_1, j_2) \neq (0,0)} \frac{(\text{Im } U)^k}{|j_1 + j_2 U|^{2k}} , \tag{2.16}$$

and $\hat{\mathcal{A}}_K$ the almost-holomorphic modular form,

$$\hat{\mathcal{A}}_K = \frac{1}{12\eta^{24}} (\hat{E}_2 E_4 E_6 + 2E_6^2 + 3E_4^3) \; . \tag{2.17}$$

The second term in (2.15) matches precisely the one-loop threshold corrections computed in [38], whereas the second line in (2.15), corresponds to $E1$ multi-instanton corrections. These are wrapping the first 2-torus, with induced worldvolume complex structure [29],

$$\mathcal{U} = \frac{j + ipU_1}{k} , \tag{2.18}$$

as depicted in figure 1. Their contribution can be also expressed as a sum over standard Hecke operators acting on (almost-holomorphic) modular invariant forms,

$$\sum_{k>j\geq 0, p>0} \frac{1}{kp} e^{-2\pi pkT_1} \left[\hat{\mathcal{A}}_f(\mathcal{U}) + \frac{1}{\pi kp} \frac{\hat{\mathcal{A}}_K(\mathcal{U})}{T_1 + \bar{T}_1} \right] = \sum_{N=1}^{\infty} e^{-2\pi NT_1} H_N \left[\hat{\mathcal{A}}_f + \frac{1}{N\pi} \frac{\hat{\mathcal{A}}_K}{T_1 + \bar{T}_1} \right] (iU_1) \tag{2.19}$$

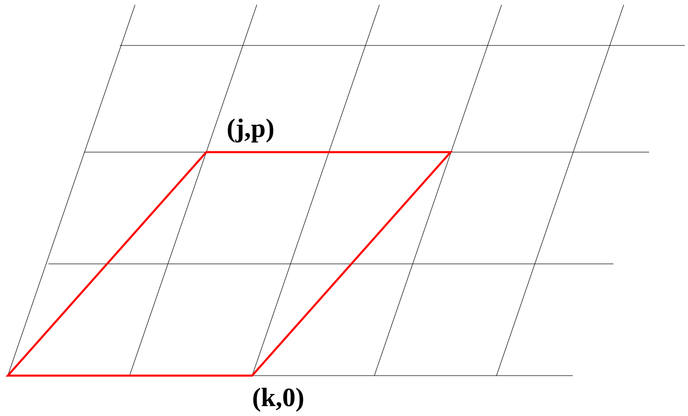

Figure 1: E1 multi-instanton wrapping the first 2-torus, with induced worldvolume complex structure $\mathcal{U}$ given in (2.18).

with,

$$H_N[\Phi](iU) = \frac{1}{N} \sum_{p>0,\ kp=N} \sum_{k>j\geq 0} \Phi\left(\frac{j+ipU}{k}\right) . \tag{2.20}$$

It is thus evident that the invariance of (2.15) under $\mathrm{SL}(2,\mathbb{Z})$ transformations of U_1, in agreement with the global symmetry preserved by the orbifold.

In order to extract from (2.15) the corrections to the effective theory, we need the Kähler metric for the D9-D9 and D9-D5 matter fields. This is given by [52, 53],

$$K_{C_k^{99}\bar{C}_k^{99}} = \frac{1}{(T_k+\bar{T}_k)(U_k+\bar{U}_k)} , \qquad K_{C_k^{95}\bar{C}_k^{95}} = \prod_{j=2,3} \frac{1}{[(T_j+\bar{T}_j)(U_j+\bar{U}_j)]^{1/2}} , \tag{2.21}$$

for $k=1,2,3$, so that,

$$\begin{aligned} \sum_r T_a(r) \log\det Z_r(M,\bar{M}) &= -16 \log[(T_1+\bar{T}_1)(U_1+\bar{U}_1)] \\ &\quad -22 \sum_{j=2,3} \log[(T_j+\bar{T}_j)(U_j+\bar{U}_j)] . \end{aligned} \tag{2.22}$$

From (1.7) and (1.8), then we read the following expressions for the corrected Kähler potential and gauge kinetic function in the effective theory,[6]

$$K = -\log(S+\bar{S}) - \sum_{i=1}^{3} \log[(T_i+\bar{T}_i)(U_i+\bar{U}_i)] + \frac{1}{2}\frac{V_{1-loop}+V_{E1}}{S+\bar{S}} , \tag{2.23}$$

$$V_{1-loop} = -\frac{4\pi}{3}\frac{E(iU_1,2)}{T_1+\bar{T}_1} , \tag{2.24}$$

$$V_{E1} = -\frac{1}{\pi} \sum_{k>j\geq 0,\ p>0} \frac{e^{-2\pi kpT_1}}{(kp)^2} \left[\frac{\hat{\mathcal{A}}_K(\mathcal{U})}{T_1+\bar{T}_1} - \frac{2ikp}{\mathcal{U}-\bar{\mathcal{U}}}\frac{E_{10}(\mathcal{U})}{\eta^{24}(\mathcal{U})}\right] + \text{ c.c.} , \tag{2.25}$$

$$f_{\mathrm{U}(16)} = S + \frac{\pi T_1}{2} - 12\log\eta(iU_1) - \frac{1}{2} \sum_{k>j\geq 0,\ p>0} \frac{e^{-2\pi kpT_1}}{kp}\mathcal{A}_f(\mathcal{U}) , \tag{2.26}$$

[6] We have performed an expansion of the logarithm in eq. (1.7) around weak coupling.

where the holomorphic modular form $\mathcal{A}_f$ is defined as in (2.13), replacing $\hat{E}_2$ by E_2. Several comments are in order. First, observe that the β-function coefficient exactly matches the field theory result. Moreover, the one-loop α' correction to the Kähler potential agrees with the expression obtained in [1, 2] by direct computation in the type I side. In our context, these corrections come from non-holomorphic terms in the contributions of degenerate orbits. Modular transformations of the T_1 modulus mix the α' corrections with the instantonic terms, in agreement with the fact that T-duality is not a symmetry of type I String Theory. Notice also that the loop correction of [12], proportional to $(\mathrm{Re}\ S)^{3/2}$, is missing. This is consistent with the fact that the internal torus has zero Euler characteristic, $\chi = 0$, for which the coefficient in front of the above correction vanishes.

From the field theory perspective, the E1 multi-instanton corrections of eq. (2.19), enter as corrections to both the Kähler potential and the holomorphic gauge kinetic function. To our knowledge, these are new corrections and their role in the low energy effective theory still has to be clarified. In section 4, we will argue that these non-perturbative corrections are general for any $\mathcal{N} = 2$ sector in orbifold compactifications where modular invariance of the target-space holds.

Finally, the presence of the first term in the heterotic threshold correction (2.15), contributing to the gauge kinetic function (2.26), may seem puzzling at first sight. Indeed, by a straightforward counting of the string coupling, this linear term in the T^2 volume modulus T_1, is expected to be a tree-level (disk) effect on the type I side. On the other hand, the T_1 modulus in the type I $\mathbb{Z}_2$ orbifold couples at tree-level only to type I D5 branes. A possible origin is the following. D9-branes in the BSGP model are fractional and therefore its gauge kinetic function should receive a contribution proportional to,

$$\sim \sum \sqrt{\det(P[G + F_2])}\ T_1 \,, \tag{2.27}$$

where the sum runs over the 16 singularities of $T^4/\mathbb{Z}_2$ and $P[\ldots]$ is the pull-back to the collapsed 2-cycle of the singularity.[7] In the orbifold limit, the volume of the 2-cycle is zero and therefore the contribution from the metric vanishes. However, as pointed out in [39], there is a non-trivial U(1) gauge bundle on the collapsed 2-cycles which, in the blow-up limit, leads together with the 8 D5-branes to the 24 instantons which are required to satisfy RR 3-form Bianchi identity, $dF_3 = \mathrm{Tr}\ R \wedge R - \mathrm{Tr}\ F_2 \wedge F_2$, in a smooth K3. One therefore expects a linear contribution to the gauge kinetic function of the D9-brane from this hidden U(1) bundle at the singularities.

2.3 $E1$ instantons

Type I String Theory and its toroidal orbifolds have E5 instantons wrapping the whole internal space and E1 instantons wrapping various two cycles, in our case instantons E1_i wrapping the T^2 torus and various two cycles inside $T^4/\mathbb{Z}_2$. Since the instantonic corrections computed in the previous section depend on the moduli of the T^2 torus, from the type I point of view they should come from E1 instantons wrapping T^2. These instantons are of two different types, depending if they sit or not at $\mathbb{Z}_2$ orbifold fixed points.

[7] We thank R. Blumenhagen for pointing out this to us.

- $E1$ *instantons at orbifold fixed points.* These instantons have unitary Chan-Paton factors, $\mathrm{U}(r)$, with neutral sector given by :

  - bosonic zero modes x_μ, $y_{1,2}$ and fermionic zero modes $\Theta^{\alpha,a}$, $\Theta^{\dot{\alpha},a}$, with $a = 1, 2$ in the adjoint representation $\mathbf{r\bar{r}}$.
  - bosonic zero modes $y_{3,4,5,6}$ and fermionic ones $\lambda^{\alpha,a}$ in the symmetric representation $\frac{\mathbf{r}(\mathbf{r}+1)}{2} + \frac{\bar{\mathbf{r}}(\bar{\mathbf{r}}+1)}{2}$.
  - fermionic zero modes $\tilde{\lambda}^{\dot{\alpha},a}$ in the antisymmetric representation $\frac{\mathbf{r}(\mathbf{r}-1)}{2} + \frac{\bar{\mathbf{r}}(\bar{\mathbf{r}}-1)}{2}$.

  Regarding the charged zero modes stretched between the instanton and the corresponding 1/2 D5-brane stuck at the singularity, we obtain:

  - bosonic zero modes $\mu^{1,2}$ from the R sector and fermionic zero modes ω^α from the NS sector, in the representation $\mathbf{r}_{-1} + \bar{\mathbf{r}}_1$, where the subscript denotes the $\mathrm{U}(1)_5$ charge.
  - bosonic zero modes $\mu'_{1,2}$ in the representation $\mathbf{r}_{+1} + \bar{\mathbf{r}}_{-1}$.

  Finally, from the E1-D9 strings, there is a bosonic zero mode ν in the representation $\mathbf{r\bar{n}} + \bar{\mathbf{r}}\mathbf{n}$.

- $E1$ *instantons off the orbifold fixed points.* These instantons have orthogonal Chan-Paton factors $\mathrm{SO}(d)$. Here we simply give their neutral sector :

  - bosonic zero modes x_μ, $y_{3,4,5,6}$ and fermionic zero modes $\Theta^{\alpha,a}$, $\Theta^{\dot{\alpha},a}$ in the representation $\frac{\mathbf{d}(\mathbf{d}+1)}{2}$.
  - bosonic zero modes $y_{1,2}$ and fermionic ones $\lambda^{\alpha,a}$, $\lambda^{\dot{\alpha},a}$ in the representation $\frac{\mathbf{d}(\mathbf{d}-1)}{2}$.

In order for the instantons to contribute to the gauge kinetic function, only four fermionic neutral zero modes should be massless (corresponding to the "goldstinos") [25]. Therefore, most of the above zero modes should be lifted by interactions. A possible qualitative picture is then the following.[8] First, notice that a $\mathrm{U}(1)$ instanton on top of a singularity correspond to a "gauge" instanton for the $\mathrm{U}(1)$ gauge theory inside the corresponding half D5-brane. These instantons are analogous to the ones discussed in [46], with the extra fermionic zero modes being lifted by couplings involving the D5-branes.[9] Therefore they should be responsible for the 1-instanton ($N = 1$) contribution in eq. (2.19). Notice however that in this case there is a Higgs branch which consists of moving the instanton out of the singularity, leading to a $\mathrm{SO}(1)$ instanton (plus its image under the orbifold). In this limit, the instanton has too many zero modes and does not correct the gauge kinetic function. Similar situations where instantons only contribute in a given locus of their moduli space have been extensively discussed in [49].

Hence, generically, for the N-instanton contribution in eq. (2.19), the moduli space of the multi-instanton contains a subspace consisting on deformations of the instanton along

[8]We thank very much A. Uranga for suggesting this picture to us and patient explanations.

[9]$\mathcal{N} = 2$ gauge instantons in String Theory orbifolds have been also extensively discussed in [47].

the $T^4/\mathbb{Z}_2$ directions. In a generic point of this space the instanton gauge group is $\mathrm{SO}(1)^N$, and the number of fermionic zero modes is too high. However, in the special locus on which all the components of the multi-instantons are on top of the same singularity, the instanton gauge group is enhanced to $\mathrm{U}(N)$ and only four zero modes survive, with the extra zero modes presumably lifted by interactions with the D5-branes.

3. The $\mathrm{SO}(q) \times \mathrm{SO}(32-q)$ freely-acting orbifold

We consider now a slightly more complex class of models, given by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ freely-acting orbifold with gauge group $\mathrm{SO}(q) \times \mathrm{SO}(32-q)$ presented in [16]. As already mentioned in the introduction, the motivation is two fold. First, to understand how non-perturbative effects are affected by the presence of a "background", breaking some of the original global symmetries. Second, to make more explicit and shed some light on the dependence of the E1 instantonic corrections on the rank of the gauge group for this class of models, as it was pointed out in [16].

In the type I side, the orbifold action on the internal coordinates is given by,

$$(x^1, x^2, x^3, x^4, x^5, x^6) \to (x^1 + 1/2, x^2, -x^3, -x^4, -x^5 + 1/2, -x^6)\,, \tag{3.1}$$
$$(x^1, x^2, x^3, x^4, x^5, x^6) \to (-x^1 + 1/2, -x^2, x^3 + 1/2, x^4, -x^5, -x^6)\,, \tag{3.2}$$
$$(x^1, x^2, x^3, x^4, x^5, x^6) \to (-x^1, -x^2, -x^3 + 1/2, -x^4, x^5 + 1/2, x^6)\ . \tag{3.3}$$

The massless $\mathcal{N} = 1$ spectrum can be read from the partition function (see [16] for details) and contains one chiral multiplet in the bifundamental representation, $(\mathbf{q}, \mathbf{32} - \mathbf{q})$. The β-function coefficient for the $\mathrm{SO}(q)$ gauge group factor then reads,

$$b_{\mathrm{SO}(q)} = 4q - 38\ . \tag{3.4}$$

Due to the discrete shifts, modular invariance of the underlying $(T^2)^3$ is broken to a subgroup of it. Moreover, the $E1$ instantons no longer appear as singlets under the orbifold action, but rather as doublets or quadruplets [16]. This kind of behavior is expected to be generic e.g. in flux compactifications, where the fluxes gauge some of the originally present symmetries and induce torsional cycles.[10]

3.1 Heterotic S-dual partition function

The partition function of the corresponding heterotic dual model was worked out in [16] for the case $q = 0$ mod 4. The action of the orbifold on the internal coordinates is given again by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. We have summarized in table 1 how each generator, f, g, h, acts on the six internal coordinates and the gauge lattice. In addition, the action of each generator is accompanied by a shift in the masses of the lattice states with (momentum, winding) = (m, n) according to,

$$(m, n) \ \xrightarrow{X}\ (m + s_X, n + s'_X)\,, \qquad X = f, g, h\ . \tag{3.5}$$

[10] A simple case are compactifications on solvmanifolds, corresponding to freely-acting orbifolds of toroidal fibrations.

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generator	x_1	x_2	x_3	x_4	x_5	x_6	SO(q)	SO($32-q$)
g	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
f	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$-$
h	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$-$

Table 1: Orbifold action on the internal coordinates and on the gauge degrees of freedom in the fermionic formulation.

Worldsheet modular invariance (or equivalently level-matching in the twisted sectors) then requires [16],

$$\begin{aligned} q = 0 \bmod 8 \qquad &\Rightarrow \qquad s_f = s_h = s_g = s'_f = s'_h = s'_g = 1/2\,, \\ q = 4 \bmod 8 \qquad &\Rightarrow \qquad s_f = s_h = s_g = s'_g = 1/2\,, s'_f = s'_h = 0\;. \end{aligned} \tag{3.6}$$

This is enough to completely determine the partition function. The concrete expressions can be found in [16].

Making use of changes of variables of the form,

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \mathcal{V}(\tau) = \int_{\mathcal{M}^{-1}(\mathcal{F})} \frac{d^2\tau}{\tau_2^2} \mathcal{V}(\mathcal{M}(\tau))\;, \tag{3.7}$$

with $\mathcal{M}$ a modular transformation, it is easy to reexpress the partition function in the more compact form,[11]

$$\begin{aligned} T = \int_{\mathcal{F}\oplus S(\mathcal{F})\oplus ST^{-1}(\mathcal{F})} \frac{d^2\tau}{\tau_2^3|\eta|^8} \frac{1}{4} \Bigg\{ &\Bigg[\frac{1}{3}(\tau_{oo}+\tau_{og}+\tau_{oh}+\tau_{of}) \frac{\hat{Z}_1\hat{Z}_2\hat{Z}_3}{|\eta|^4} + \\ &+(\tau_{oo}+\tau_{og}-\tau_{oh}-\tau_{of})(-1)^{m_1+n_1}\hat{Z}_1 \left|\frac{4\eta^2}{\vartheta_2^2}\right|^2 \Bigg] (\overline{\chi_o+\chi_v}) + \\ &+\Bigg[(\tau_{oo}-\tau_{og}+\tau_{oh}-\tau_{of})(-1)^{m_3+n_3+\frac{qn_3}{4}}\hat{Z}_3 + \\ &+(\tau_{oo}-\tau_{og}-\tau_{oh}+\tau_{of})(-1)^{m_2+n_2+\frac{qn_2}{4}}\hat{Z}_2\Bigg] \left|\frac{4\eta^2}{\vartheta_2^2}\right|^2 (\overline{\chi_o-\chi_v}) \Bigg\}\,, \end{aligned} \tag{3.8}$$

where the characters χ_o and χ_v are given in the fermionic formulation of the gauge degrees of freedom by,

$$\chi_o = O_{32-q}O_q + C_{32-q}C_q \quad , \quad \chi_v = V_{32-q}V_q + S_{32-q}S_q\,, \tag{3.9}$$

with O_r, V_r, S_r and C_r the standard SO(r) affine characters. The lattice sums with a sign insertion are given by,

$$\begin{aligned} (-1)^{m_1+hn_1}\hat{Z}_i = \frac{\text{Re}\;T_i}{\tau_2} &\sum_{n_1,\ell_1,n_2,\ell_2} (-1)^{hn_1\ell_1} \\ &\times\exp\left[-2\pi T_i \det(A) - \frac{\pi(\text{Re}\;T_i)}{\tau_2(\text{Re}\;U_i)}\left|\begin{pmatrix}1 & iU_i\end{pmatrix} A \begin{pmatrix}\tau \\ -1\end{pmatrix}\right|^2\right]\,, \end{aligned} \tag{3.10}$$

[11]These changes of variables are of course not unique. We could have equally chosen a different set of modular transformations $\mathcal{M}$ (coset representatives), leading to a different integrand and integration region.

and,

$$A = \begin{pmatrix} n_1 & \ell_1 + \frac{1}{2} \\ n_2 & \ell_2 \end{pmatrix} \tag{3.11}$$

The whole KK spectrum precisely matches the corresponding one on the type I S-dual side, whereas the massive winding states and the massive twisted spectra are, as expected, quite different. It should be also noticed that while the KK spectra are actually the same for the two cases, $q = 0$ and $q = 4$ (mod 8), they are very different in the massive winding sector. We refer the interested reader to [16] for the concrete expressions of the partition functions in the type I S-dual side and other details.

3.2 Perturbative and non-perturbative corrections

Starting with the partition function (3.8) and proceeding in the same way as we did with the BSGP orbifold, it can be shown that the threshold corrections to the physical gauge couplings (c.f. eq. (1.1)) are given in this case by,

$$\begin{aligned}\Lambda_{\mathrm{SO}(q)} &\equiv \frac{b_{\mathrm{SO}(q)}}{4}\log\frac{M_s^2}{\mu^2} + \frac{\Delta_{\mathrm{SO}(q)}}{4} \\ &= -\frac{1}{4}\int_{\mathcal{F}_{\Gamma_0(2)}} \frac{d^2\tau}{\tau_2}\left[(-1)^{m_1+n_1}\hat{Z}_1\hat{\mathcal{A}}^{[0,1]}_{f,1} + \sum_{r=2,3}(-1)^{m_r+n_r+\frac{qn_r}{4}}\hat{Z}_r\hat{\mathcal{A}}^{[0,1]}_{f,2}\right],\end{aligned} \tag{3.12}$$

where,

$$\hat{\mathcal{A}}^{[0,1]}_{f,1}(\tau) = \frac{\vartheta_3^2\vartheta_4^2 E_4(\hat{E}_2E_4 - E_6)}{12\eta^{24}} \tag{3.13}$$

$$\hat{\mathcal{A}}^{[0,1]}_{f,2}(\tau) = \frac{\vartheta_3^2\vartheta_4^2}{24\eta^{24}}\left[\vartheta_3^{q/2}\vartheta_4^{16-q/2}(\hat{E}_2 + \vartheta_2^4 - \vartheta_4^4) + \vartheta_4^{q/2}\vartheta_3^{16-q/2}(\hat{E}_2 - \vartheta_2^4 - \vartheta_3^4)\right] \tag{3.14}$$

The details can be found in appendix A.2. In order to perform this integral, notice that the integration region,

$$\mathcal{F}_{\Gamma_0}(2) \equiv \mathcal{F} \oplus S(\mathcal{F}) \oplus ST^{-1}(\mathcal{F}), \tag{3.15}$$

which we have represented in figure 2, corresponds to the fundamental domain of the congruence subgroup $\Gamma_0(2) \subset \mathrm{SL}(2,\mathbb{Z})$. This consists of the modular matrices of the form [43, 44],

$$\begin{pmatrix} 2a+1 & b \\ 2c & 2d+1 \end{pmatrix} \tag{3.16}$$

The generators of $\Gamma_0(2)$ are T and ST^2S. Under these, $\hat{\mathcal{A}}^{[0,1]}_{f,2}$ transforms as,

$$\hat{\mathcal{A}}^{[0,1]}_{f,2} \xrightarrow{T} \hat{\mathcal{A}}^{[0,1]}_{f,2}, \quad \hat{\mathcal{A}}^{[0,1]}_{f,2} \xrightarrow{ST^2S} (-1)^{q/4}\hat{\mathcal{A}}^{[0,1]}_{f,2}, \tag{3.17}$$

whereas $\hat{\mathcal{A}}^{[0,1]}_{f,1}$ keeps invariant. We can therefore classify the matrices (3.11) in orbits under $\Gamma_0(2)$ in order to unfold the integral (3.12), similarly to what we did for the BSGP

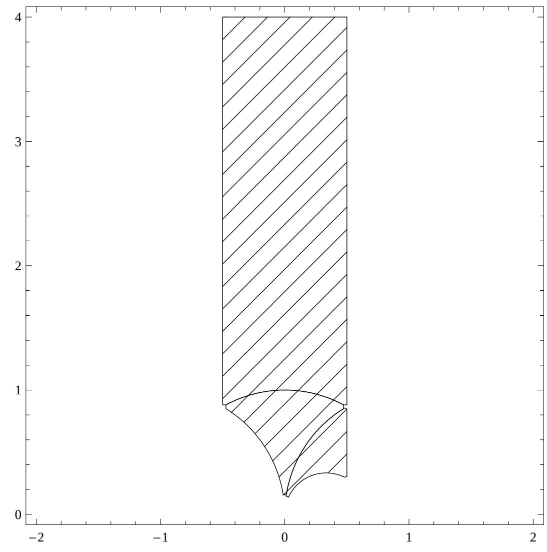

Figure 2: Representation of $\mathcal{F}_{\Gamma_0(2)}$, the fundamental domain for $\Gamma_0(2)$.

model.[12] There are four kinds of orbits (three non-degenerate and one degenerate), whose representatives can be taken to be,

1. *Degenerate orbits:*
$$\begin{pmatrix} 0 & j+\frac{1}{2} \\ 0 & p \end{pmatrix},$$
with $(j,p) \sim (-j-1,-p)$ and $AV = AV'$ iff $V = T^n V'$ for some integer n and $V, V' \in \Gamma_0(2)$.

2. *Non-degenerate orbits:*
$$\text{I:} \quad \begin{pmatrix} k & j+\frac{1}{2} \\ 0 & p \end{pmatrix}, \qquad \text{II:} \quad \begin{pmatrix} j & -k-\frac{1}{2} \\ p & 0 \end{pmatrix}, \qquad \text{III:} \quad \begin{pmatrix} j-k & -k-\frac{1}{2} \\ p & 0 \end{pmatrix},$$
with $k > j \geq 0$, $p \neq 0$ and $AV = AV'$ iff $V = V'$, for $V, V' \in \Gamma_0(2)$.

We can therefore unfold (3.12) into four integrals corresponding to the above representatives. The details are again relegated to the appendix. Putting all pieces together we

[12] One could worry about the sign in the transformation of $\hat{\mathcal{A}}^{[0,1]}_{f,2}$ under ST^2S, for $q = 4 \bmod 8$. However, this is automatically cancelled by the transformation of the lattice sum,

$$(-1)^{m_r+n_r+\frac{qn_r}{4}}\hat{Z}_r \xrightarrow{ST^2S} (-1)^{q/4}(-1)^{m_r+n_r+\frac{qn_r}{4}}\hat{Z}_r \tag{3.18}$$

as required by modular invariance of (3.12). Alternatively, we could have performed an extra change of variables in (3.12) and reexpressed it as an integral over the fundamental domain of $\Gamma_0(4) \subset \Gamma_0(2)$, given by modular matrices of the form,

$$\begin{pmatrix} 2a+1 & b \\ 4c & 2d+1 \end{pmatrix}, \tag{3.19}$$

obtaining the same final result.

obtain,

$$
\begin{aligned}
\Lambda_{\mathrm{SO}(q)} = & -\frac{\pi}{2}\left[5E_{1/2}(iU_1,1)+\frac{q-17}{3}\sum_{r=2,3}E_{1/2}(iU_r,1)\right]+ \\
& +\frac{\pi}{360}\left[124\frac{E_{1/2}(iU_1,2)}{T_1+\bar{T}_1}+\frac{1}{2}\sum_{r=2,3}(q^2-32q+248)\frac{E_{1/2}(iU_r,2)}{T_r+\bar{T}_r}\right]- \\
& -\frac{1}{4}\sum_{k>j\geq 0,\ p>0}\ \sum_{[h,g]}\frac{(-1)^{gk+hj}}{pk_h}\Bigg[e^{-2\pi pk_h T_1}\left(\hat{\mathcal{A}}^{[h,g]}_{f,1}\left(\mathcal{U}_1^{[h,g]}\right)+\frac{1}{\pi pk_h}\frac{\hat{\mathcal{A}}^{[h,g]}_{K,1}\left(\mathcal{U}_1^{[h,g]}\right)}{T_1+\bar{T}_1}\right)+ \\
& +\sum_{r=2,3}(-1)^{(hj+gk)\frac{q}{4}}e^{-2\pi pk_h T_r}\left(\hat{\mathcal{A}}^{[h,g]}_{f,2}\left(\mathcal{U}_r^{[h,g]}\right)+\frac{1}{\pi pk_h}\frac{\hat{\mathcal{A}}^{[h,g]}_{K,2}\left(\mathcal{U}_r^{[h,g]}\right)}{T_r+\bar{T}_r}\right)\Bigg]+\text{c.c.}\,,
\end{aligned}
\tag{3.20}
$$

where $\hat{\mathcal{A}}^{[h,g]}_{f,i}$ and $\hat{\mathcal{A}}^{[h,g]}_{K,i}$, $i=1,2$, are given in appendix A.2.1, $k_h \equiv k+\frac{h}{2}$, and the shifted non-holomorphic Eisenstein series, $E_{1/2}(U,k)$, are defined as,

$$
E_{1/2}(U,k) \equiv \frac{1}{\zeta(2k)}\sum_{j_1,j_2}\frac{(\mathrm{Im}\ U)^k}{|j_1+j_2U+1/2|^{2k}}\,, \tag{3.21}
$$

In particular [16],

$$
E_{1/2}(iU,1) = -\frac{3}{\pi}(\log|\vartheta_2(iU)|^4+\pi\mathrm{Re}\ U+\log[(\mathrm{Re}\ U)(\mathrm{Re}\ T)\mu^2])\ +\ \text{const.}\,. \tag{3.22}
$$

The sum of $[h,g]$ in (3.20) extends over $[1,0]$, $[0,1]$ and $[1,1]$, labelling the three types of $E1_r$ multi-instantons contributing to (3.12). The induced complex structure on their worldvolume is given by,

$$
\mathcal{U}_r^{[h,g]} = \frac{j+ipU_r+g/2}{k+h/2}\,, \tag{3.23}
$$

corresponding to instantons wrapping the torsional cycles of the twisted cohomology, as illustrated in figure 3, or alternatively, multi-instantons with discrete Wilson lines $(\alpha,\beta)\in\{(0,\frac{1}{2}),(1,0),(1,\frac{1}{2})\}$ [24].

Subtracting the gauge group dependent part, along the lines of eqs. (1.7) and (1.8), we obtain the following corrections to the effective Kähler potential and gauge kinetic function,

$$
K = -\log(S+\bar{S})-\sum_{i=1}^{3}\log[(T_i+\bar{T}_i)(U_i+\bar{U}_i)]+\frac{1}{2}\sum_{i=1}^{3}\frac{V^i_{1-loop}+V^i_{E1}}{S+\bar{S}}\,, \tag{3.24}
$$

$$
V^1_{1-loop} = \frac{62\pi}{45}\frac{E_{1/2}(iU_1,2)}{T_1+\bar{T}_1}\,, \tag{3.25}
$$

$$
V^r_{1-loop} = \frac{\pi}{180}(q^2-32q+248)\frac{E_{1/2}(iU_r,2)}{T_r+\bar{T}_r}\,, \tag{3.26}
$$

$$
\begin{aligned}
V^1_{E1} = & -\sum_{k>j\geq 0,\ p>0}\ \sum_{[h,g]}\frac{(-1)^{gk+hj}}{p^2k_h^2}e^{-2\pi pk_h T_1} \\
& \times\left[\frac{\hat{\mathcal{A}}^{[h,g]}_{K,1}\left(\mathcal{U}_1^{[h,g]}\right)}{T_1+\bar{T}_1}-\frac{i\pi pk_h}{\mathcal{U}_1^{[h,g]}-\bar{\mathcal{U}}_1^{[h,g]}}\chi_1\begin{bmatrix}h\\g\end{bmatrix}(\mathcal{U}_1^{[h,g]})\right]\ +\ \text{c.c.}
\end{aligned}
\tag{3.27}
$$

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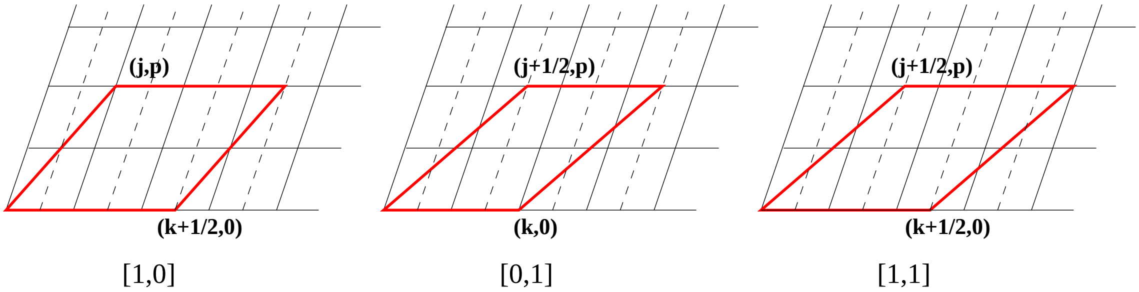

Figure 3: The three possible types of $E1_r$ multi-instantons, $[h,g] = \{[1,0],[0,1],[1,1]\}$, wrapping torsional cycles in the r-th 2-torus in the $\mathrm{SO}(q) \times \mathrm{SO}(32-q)$ model, with induced worldvolume complex structure $\mathcal{U}_r^{[h,g]}$ given in eq. (3.23). The continues lines represent the lattice of the underlying 2-torus. The orbifold generator reversing the transverse coordinates to the instanton, shifts the lattice to the dashed one.

$$
\begin{aligned}
V_{E1}^r = & - \sum_{k>j\geq 0,\ p>0} \sum_{[h,g]} \frac{(-1)^{(hj+gk)\left(\frac{q}{4}+1\right)}}{p^2 k_h^2} e^{-2\pi p k_h T_r} \qquad (3.28)\\
& \times \left[\frac{\hat{\mathcal{A}}_{K,2}^{[h,g]}\left(\mathcal{U}_r^{[h,g]}\right)}{T_r + \bar{T}_r} - \frac{i\pi p k_h}{\mathcal{U}_r^{[h,g]} - \bar{\mathcal{U}}_r^{[h,g]}} \chi_2 \begin{bmatrix} h \\ g \end{bmatrix} (\mathcal{U}_r^{[h,g]}) \right] + \text{ c.c. },\\
f_{\mathrm{SO}(q)} = & \ S + \frac{15\pi}{2} U_1 + 30 \log\, \vartheta_2(iU_1) + (q-17) \sum_{r=2,3} \left[\frac{\pi}{2} U_r + 2 \log\, \vartheta_2(iU_r)\right] - \qquad (3.29)\\
& - \frac{1}{2} \sum_{k>j\geq 0,\ p>0} \sum_{[h,g]} \frac{(-1)^{gk+hj}}{p k_h} \Bigg[e^{-2\pi p k_h T_1} \mathcal{A}_{f,1}^{[h,g]}(\mathcal{U}_1^{[h,g]}) + \\
& \qquad + \sum_{r=2,3} (-1)^{(hj+gk)\frac{q}{4}} e^{-2\pi p k_h T_r} \mathcal{A}_{f,2}^{[h,g]}(\mathcal{U}_r^{[h,g]}) \Bigg] ,
\end{aligned}
$$

where $\mathcal{A}_{f,i}^{[h,g]}$ is defined as $\hat{\mathcal{A}}_{f,i}^{[h,g]}$, but replacing $\hat{E}_2$ by E_2, and we have introduced the notation,

$$
\begin{aligned}
\chi_1 \begin{bmatrix} h \\ g \end{bmatrix} (\tau) &\equiv \frac{4(\chi_o + \chi_v)}{\eta^2 \vartheta \begin{bmatrix} 1-h \\ 1-g \end{bmatrix}^2} , &\qquad \chi_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\tau) &\equiv \frac{\chi_o - \chi_v}{\eta^8} \vartheta_3^2 \vartheta_4^2 ,\\
\chi_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\tau) &\equiv \chi_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (S\tau) , &\qquad \chi_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\tau) &\equiv \chi_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} (ST^{-1}\tau) .
\end{aligned}
$$

Several comments are in order. First, notice that the field theory result for the β-function coefficient, (3.4), is correctly reproduced. The overall structure of the non-perturbative and loop corrections is very similar to the ones in the BSGP orbifold, but the standard Eisenstein series and Hecke operators are replaced by the corresponding automorphic forms of $\Gamma_0(2)$. Moreover, there is a non-trivial dependence of the non-perturbative dynamics on the rank of the gauge group, through the phases $\exp[i\pi(hj+gk)\frac{q}{4}]$. These would explain why in the heterotic side the orbifold action on the winding modes is very different, depending on the value of q (c.f. eq. (3.6)). This behavior may resemble in spirit the more familiar situation of ordinary gauge theory instantons, where their contributions are often subjected to constraints depending on the ranks of the gauge group.

By a direct inspection of (3.20), it is easy to check the fact that instantonic corrections are gauge-group independent when they come from instantons which are left invariant by the orbifold operations acting trivially on the gauge degrees of freedom; whereas they are gauge-group dependent if they come from instantons left invariant by the orbifold operations which act non-trivially on the gauge degrees of freedom.

Finally, let us mention the possibility of additional non-perturbative corrections coming from purely $\mathcal{N}=1$ sectors, not considered here. Precisely, in [24] it was argued for the $q=32$ case, the presence of extra non-perturbative contributions to the gauge thresholds, due to the combined effect of multi-instantons wrapping different cycles of the internal space.

4. Universality of $\mathcal{N}=2$ corrections

The importance of global symmetries in determining the expression of non-perturbative and α' corrections which come from BPS states has been pointed out very often in the literature [28]. The results in the previous sections, based on the S-duality map, reveal that the string loop and $E1$ multi-instanton effects coming from $\mathcal{N}=2$ subsectors of the theory arise in terms of non-holomorphic Eisenstein series and Hecke operators relevant to the global symmetry preserved by the orbifold. In this section, we elaborate on a certain "universality" of the $\mathcal{N}=2$ corrections computed in the BSGP orbifold. Similar aspects have been discussed in the context of α' corrections to the gauge couplings in heterotic compactifications [34, 36, 20, 37].

Precisely, we would like to consider toroidal orbifold compactifications on which the orbifold action, $\mathbb{G}$, contains some subgroup, $\mathbb{G}^i$, leaving unrotated a given complex plane. The contribution of these sectors to the threshold corrections to the physical gauge couplings can be expressed as,

$$\Lambda_a = -\frac{1}{8}\sum_i \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \hat{Z}_i \hat{\mathcal{A}}^a_{f,i}\,, \tag{4.1}$$

where the sum runs over the disjoint union of $\mathcal{N}=2$ subsectors, each leaving invariant a single complex plane, and the gauge group is given by a product $G=\prod_a G_a$. The lattice sums, $\hat{Z}_i$, are given in eq. (2.3), where T_i and U_i are now the Kähler and complex structure moduli of the corresponding unrotated complex plane. Moreover, $\hat{\mathcal{A}}^a_{f,i} \sim M^a_i/\eta^{24}$, with M^a_i an almost holomorphic modular form of degree 24.

The space of holomorphic forms of degree 24 is a vector space of dimension 2, engendered by the Eisenstein series E_6^2 and E_4^3 [43, 44]. If we also allow for almost holomorphic modular forms, we have to include in addition $\hat{E}_2 E_{10}$.[13] Hence M^a_i is in general determined by three coefficients, which usually can be obtained from the low energy spectrum. More precisely, imposing the absence of tachyons in the spectrum, we obtain,

$$\hat{\mathcal{A}}^a_{f,i} = 2b^a_i + \frac{\gamma_i}{20\eta^{24}}\left[D_{10}E_{10} - 528\eta^{24}\right]\ , \tag{4.2}$$

[13] We could also think about including terms with higher powers of $\hat{E}_2$, e.g. $\hat{E}_2^2 E_4^2$. However, these terms are forbidden by $\mathcal{N}=2$ supersymmetry [29].

where b_i^a is the β-function coefficient of the $\mathcal{N}=2$ gauge theory associated to a would-be $T^6/\mathbb{G}^i$ orbifold, γ_i is a model dependent (but gauge group independent) coefficient to be determined, and we have made use of the identities,

$$D_{10}E_{10} = \frac{2}{3}E_6^2 + E_4^3 - \frac{5}{3}\hat{E}_2 E_{10}\,, \qquad \eta^{24} = \frac{1}{2^6\cdot 3^3}[E_4^3 - E_6^2]\ . \tag{4.3}$$

Proceeding as in section 2.2 we get,

$$\begin{aligned}\Lambda_a = \sum_i \Bigg\{ & \frac{\pi(b_i^a + 6\gamma_i)}{12}\text{Re } T_i \ + \ \frac{\pi\gamma_i}{3}\frac{E(iU_i,2)}{T_i+\bar{T}_i} \\ & -\frac{1}{4}\left(\sum_{k>j\geq 0, p>0}\frac{1}{kp}e^{-2\pi pkT_i}\left[\hat{\mathcal{A}}^a_{f,i}(\mathcal{U}_i) - \frac{\gamma_i}{\pi kp}\frac{\hat{\mathcal{A}}_K(\mathcal{U}_i)}{T_i+\bar{T}_i}\right] \ + \ \text{c.c.}\right) \\ & -\frac{b_i^a}{4}\left(\log|\eta(iU_i)|^4 + \log[(\text{Re } U_i)(\text{Re } T_i)\mu^2]\right)\Bigg\}\,, \end{aligned} \tag{4.4}$$

with $\hat{\mathcal{A}}_K$ and $\mathcal{U}_i$ defined in (2.17) and (2.18), respectively. From this expression, we can then extract the corrected Kähler potential and gauge kinetic functions of the effective theory, as we did in previous sections, obtaining

$$K = -\log(S+\bar{S}) - \sum_i \left\{\log[(T_i+\bar{T}_i)(U_i+\bar{U}_i)] + \frac{1}{2}\frac{V^i_{1-loop} + V^i_{E1}}{S+\bar{S}}\right\} + \cdots\,, \tag{4.5}$$

$$V^i_{1-loop} = \frac{4\pi\gamma_i}{3}\frac{E(iU_i,2)}{T_i+\bar{T}_i}\,, \tag{4.6}$$

$$V^i_{E1} = \frac{\gamma_i}{\pi}\sum_{k>j\geq 0,\ p>0}\frac{e^{-2\pi kpT_i}}{(kp)^2}\left[\frac{\hat{\mathcal{A}}_K(\mathcal{U}_i)}{T_i+\bar{T}_i} - \frac{2ikp}{\mathcal{U}_i - \bar{\mathcal{U}}_i}\frac{E_{10}(\mathcal{U}_i)}{\eta^{24}(\mathcal{U}_i)}\right] \ + \ \text{c.c.}\,, \tag{4.7}$$

$$f_a = S + \sum_i \left\{\frac{\pi(b_i^a + 6\gamma_i)T_i}{12} - b_i^a \log\, \eta(iU_i) - \frac{1}{2}\sum_{k>j\geq 0,\ p>0}\frac{e^{-2\pi kpT_i}}{kp}\mathcal{A}^a_{f,i}(\mathcal{U}_i)\right\} + \cdots\,, \tag{4.8}$$

where the dots refer to possible additional corrections from other sectors. The interpretation of these terms is similar to the one discussed in sections 2.2 and 2.3.

Notice that these expressions in principle also apply in orbifolds where the heterotic S-dual description is unknown, and therefore our technique in principle no longer applies. It would be very interesting to obtain the general formula for the $\mathcal{N}=2$ corrections by direct computation in the type I orbifold, and to see whether there is agreement with our conjectured expression.

5. Concluding remarks

In the present paper we explicitly computed the $E1$ instantonic corrections to the gauge kinetic function f and to the Kähler potential K in $\mathcal{N}=2$ and $\mathcal{N}=1$ type I string vacua which have known heterotic S-duals. We showed that one-loop threshold corrections to gauge couplings in the heterotic dual encode one-loop and instantonic corrections for both

f and K on the type I side, whereas the corresponding direct one-loop type I threshold correction misses the one-loop correction to the Kähler potential, computed by other methods in [1] and [2]. We gave arguments based on target-space modular invariance on universality properties of instantonic corrections in $\mathcal{N}=2$ vacua. It is clear however that our results apply to the much larger class of models of $\mathcal{N}=1$ type I models with $\mathcal{N}=2$ subsectors, like for example the $\mathbb{Z}_6$, $\mathbb{Z}_6'$, $\mathbb{Z}_8$ or $\mathbb{Z}_{12}$ type I orbifolds.

We performed a similar computation in dual pairs in compactifications on smooth Calabi-Yau spaces which have an exact CFT description, based on a recently worked out class of freely-acting S-dual pairs [16]. We showed that even if the heterotic duals of perturbatively connected type I models have different orbifold actions in the twisted (winding) sector, the S-duality maps correctly heterotic α' corrections into type I instantonic corrections. As a byproduct, we also checked the intuitively obvious statement that instantonic corrections are gauge-group independent if coming from instantons left invariant by orbifold operations acting trivially on the gauge degrees of freedom, whereas they are gauge-group dependent if coming from instantons left invariant by orbifold operations acting non-trivially on the gauge degrees of freedom.

As already argued in [16], it is clear that whereas our discussion was focused on multi-instantonic corrections to the gauge kinetic function and the Kähler potential, similar multi-instanton corrections are expected to occur for the superpotential. A simple argument can be given in the case (explicitly realized by the string construction of [16]) where non-perturbative gauge (E5 instantonic) effects occur on D9-branes, leading to a superpotential,

$$W_{np} \; = e^{-b(S+f_1(U_i)+f_{np}(U_i,T_i))} \; = \; \sum_n c_n(U_i) \; e^{-2\pi n T} \; e^{-bS} \tag{5.1}$$

Whereas non-perturbative corrections to the superpotential are well-known to play a crucial role in moduli stabilization [10], we expect that instantonic corrections to the Kähler potential may play also an important role in some scenarios of moduli stabilization, for example in the large-volume scenario [13]. Moreover, the instantonic corrections to the gauge kinetic function are expected to modify the gauge couplings and gaugino masses, and in particular may become relevant in concrete phenomenological models.

Another interesting direction which our paper has left partially open is the detailed type I microscopic derivation of the multi-instanton effects obtained here from S-duality, which should involve in an important way the lifting of fermionic zero-modes by instanton interactions along the lines of [48, 49].

It would be, finally, very instructive to perform similar studies in the S-dual pairs of $\mathcal{N}=1$ orbifold models conjectured in [50, 51] and learn more about non-perturbative dynamics of both sides using α' corrections on the heterotic side and instantonic computations on the type I side.

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A. Details on the computations

A.1 The BSGP model

A.1.1 Elliptic genera

In order to express the elliptic genus in terms of ordinary modular forms we use the following relations,

$$\vartheta^4\begin{bmatrix}0\\ \pm1/2\end{bmatrix} = \frac{1}{2}\vartheta_3\vartheta_4(\vartheta_3^2+\vartheta_4^2)\,, \qquad \vartheta^4\begin{bmatrix}1\\ \pm1/2\end{bmatrix} = \frac{\vartheta_2^3\eta^3}{\vartheta_3^2+\vartheta_4^2}\,, \tag{A.1}$$

$$\vartheta^4\begin{bmatrix}\pm1/2\\ 0\end{bmatrix} = \frac{1}{2}\vartheta_2\vartheta_3(\vartheta_2^2+\vartheta_3^2)\,, \qquad \vartheta^4\begin{bmatrix}\pm1/2\\ 1\end{bmatrix} = -\frac{\vartheta_4^3\eta^3}{\vartheta_2^2+\vartheta_3^2}\,, \tag{A.2}$$

$$\vartheta^4\begin{bmatrix}\pm1/2\\ \pm1/2\end{bmatrix} = \frac{1}{2}\vartheta_2\vartheta_4(\vartheta_2^2-i\vartheta_4^2)\,, \qquad \vartheta^4\begin{bmatrix}\pm1/2\\ \mp1/2\end{bmatrix} = \frac{\vartheta_3^3\eta^3}{\vartheta_2^2-i\vartheta_4^2}\,, \tag{A.3}$$

and,

$$\frac{\vartheta''\begin{bmatrix}0\\ \pm1/2\end{bmatrix}}{\vartheta\begin{bmatrix}0\\ \pm1/2\end{bmatrix}} = \frac{i\pi^3}{3}(4E_2+\vartheta_3^4-6\vartheta_3^2\vartheta_4^2+\vartheta_4^4)\ , \tag{A.4}$$

$$\frac{\vartheta''\begin{bmatrix}1\\ \pm1/2\end{bmatrix}}{\vartheta\begin{bmatrix}1\\ \pm1/2\end{bmatrix}} = \frac{i\pi^3}{3}(4E_2+\vartheta_3^4+6\vartheta_3^2\vartheta_4^2+\vartheta_4^4)\ , \tag{A.5}$$

$$\frac{\vartheta''\begin{bmatrix}\pm1/2\\ 0\end{bmatrix}}{\vartheta\begin{bmatrix}\pm1/2\\ 0\end{bmatrix}} = \frac{i\pi^3}{3}\left(4E_2+4\vartheta_3^4+\frac{(\vartheta_2^2-5\vartheta_3^2)\vartheta_4^4}{\vartheta_2^2+\vartheta_3^2}\right)\ , \tag{A.6}$$

$$\frac{\vartheta''\begin{bmatrix}\pm1/2\\ 1\end{bmatrix}}{\vartheta\begin{bmatrix}\pm1/2\\ 1\end{bmatrix}} = \frac{i\pi^3}{3}\left(4E_2-8\vartheta_3^4+\frac{(\vartheta_2^2+7\vartheta_3^2)\vartheta_4^4}{\vartheta_2^2+\vartheta_3^2}\right)\ , \tag{A.7}$$

$$\frac{\vartheta''\begin{bmatrix}\pm1/2\\ \pm1/2\end{bmatrix}}{\vartheta\begin{bmatrix}\pm1/2\\ \pm1/2\end{bmatrix}} = \frac{i\pi^3}{3}\left(4E_2-8\vartheta_4^4+7\vartheta_3^4-\frac{6\vartheta_2^2\vartheta_3^4}{\vartheta_2^2-i\vartheta_4^2}\right)\ , \tag{A.8}$$

$$\frac{\vartheta''\begin{bmatrix}\pm1/2\\ \mp1/2\end{bmatrix}}{\vartheta\begin{bmatrix}\pm1/2\\ \mp1/2\end{bmatrix}} = \frac{i\pi^3}{3}\left(4E_2+4\vartheta_4^4-5\vartheta_3^4+\frac{6\vartheta_2^2\vartheta_3^4}{\vartheta_2^2-i\vartheta_4^2}\right)\ . \tag{A.9}$$

Then it is possible to show that,

$$A_1 \equiv 2\left(\bar\Gamma\begin{bmatrix}0\\1\end{bmatrix}\vartheta_3^2\vartheta_4^2+\bar\Gamma\begin{bmatrix}1\\0\end{bmatrix}\vartheta_2^2\vartheta_3^2+\bar\Gamma\begin{bmatrix}1\\1\end{bmatrix}\vartheta_2^2\vartheta_4^2\right) = \frac{2E_4E_6}{\eta^{16}} = \frac{2E_{10}}{\eta^{16}}\,, \tag{A.10}$$

$$A_2 \equiv \partial_{\nu_1}^2 A_1 = -\frac{2\pi^2}{3\eta^{16}}\left(E_2E_4E_6-\frac{5}{12}E_6^2-\frac{7}{12}E_4^3\right), \tag{A.11}$$

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where ∂_{ν_1} acts on the first SO(2) character in $\bar{\Gamma}\begin{bmatrix} h \\ g \end{bmatrix}$, with affine parameter ν_1. Therefore,

$$\hat{\mathcal{A}}_f \equiv -\frac{1}{8\pi\bar{\eta}^8}\left(\frac{A_1}{\tau_2}+\frac{A_2}{\pi}\right) = \frac{1}{12\eta^{24}}\left(\hat{E}_2 E_{10} - \frac{5}{12}E_6^2 - \frac{7}{12}E_4^3\right) = -24 + \frac{60}{\pi\tau_2} + \cdots \quad \text{(A.12)}$$

The other modular form that we will need in the computation of the thresholds is,

$$\hat{\mathcal{A}}_K \equiv \frac{1}{4\pi}\left(i\partial_\tau - \frac{1}{\tau_2}\right)\frac{E_{10}}{\eta^{24}} \quad \text{(A.13)}$$

Taking into account that $E_{10} = E_4 E_6$ and,

$$\partial_\tau E_4 = -\frac{2\pi i}{3}(E_6 - E_2 E_4)\,, \qquad \partial_\tau E_6 = -\pi i (E_4^2 - E_2 E_6)\,, \quad \text{(A.14)}$$

we then obtain,

$$\hat{\mathcal{A}}_K = \frac{1}{12\eta^{24}}(\hat{E}_2 E_4 E_6 + 2E_6^2 + 3E_4^3)\ . \quad \text{(A.15)}$$

A.1.2 Zero orbit

For the zero orbit we have the contribution,

$$\begin{aligned}\Lambda_0 &= -\frac{\text{Re } T_1}{8}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\hat{\mathcal{A}}_f = \\ &= -\frac{\text{Re } T_1}{96}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\left[\hat{E}_2(e^{-2\pi i\tau} - 240 + \cdots) - e^{-2\pi i\tau} - 24 + \cdots\right]\end{aligned} \quad \text{(A.16)}$$

Making use of the formula [45],

$$\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}(\hat{E}_2)^r(c_{-1}e^{-2\pi i\tau} + c_0 + \cdots) = \frac{\pi}{3(r+1)}[c_0 - 24(r+1)c_{-1}]\,, \quad \text{(A.17)}$$

we get,

$$\Lambda_0 = \frac{\pi}{2}(\text{Re } T_1)\ . \quad \text{(A.18)}$$

A.1.3 Degenerate orbits

In this case we have to compute the contribution,

$$\Lambda_d = -\frac{\text{Re } T_1}{8}\int_{\mathcal{F}_T}\frac{d^2\tau}{\tau_2^2}\left(\frac{60}{\pi\tau_2} - 24 + \mathcal{O}(e^{2\pi i\tau})\right)\sum_{j,p}\exp\left[-\frac{\pi\text{Re } T_1}{\tau_2\text{Re } U_1}|j + ipU_1|^2\right] \quad \text{(A.19)}$$

where the integration region, $\mathcal{F}_T$, corresponds to the upper band $\{|\tau_1| < 1/2,\ \tau_2 > 0\}$. This can be done using the formula [29],

$$\int_{\mathcal{F}_T}\frac{d^2\tau}{\tau_2^{1+r}}\sum_{j,p}\exp\left[-\frac{\pi\text{Re } T}{\tau_2\text{Re } U}|j + ipU|^2\right] = \frac{2\Gamma(r)\zeta(2r)}{(\pi\text{Re } T)^r}E(iU, r)\ . \quad \text{(A.20)}$$

Taking into account that,

$$E(iU, 1) = -\frac{3}{\pi}(\log|\eta(iU)|^4 + \log[(\text{Re } T)(\text{Re } U)]\mu^2)\ +\ \text{const.}\,, \quad \text{(A.21)}$$

with μ^2 the infrared regulator and "const." a renormalization scheme dependent constant which we will disregard in what follows, we obtain,

$$\Lambda_d = -3(\log|\eta(iU_1)|^4 + \log[(\text{Re } T_1)(\text{Re } U_1)\mu^2]) - \frac{\pi}{3}\frac{E(iU_1, 2)}{T_1 + \bar{T}_1}\ . \quad \text{(A.22)}$$

A.1.4 Non-degenerate orbits

Finally, for non-degenerate orbits we need to compute,

$$\Lambda_{nd} = -\frac{\text{Re } T_1}{4}\int_{\mathbb{C}^+}\frac{d^2\tau}{\tau_2^2}\sum_{k>j\geq 0,\ p\neq 0}\sum_n\left(d_1(n)-\frac{d_2(n)}{4\pi\tau_2}\right)\ \times$$
$$\times\ e^{2\pi i\tau n}\text{exp}\left[-2\pi T_1 kp - \frac{\pi\text{Re } T_1}{\tau_2\text{Re } U_1}|-j-iU_1p+k\tau|^2\right], \qquad \text{(A.23)}$$

where we have expanded,

$$\mathcal{A}_f = \sum_n d_1(n)e^{2\pi i n\tau}\,, \qquad \frac{E_{10}}{\eta^{24}} = \sum_n d_2(n)e^{2\pi i n\tau} \qquad \text{(A.24)}$$

Performing first the integration on τ_1,

$$\Lambda_{nd} = -\frac{[(\text{Re } T_1)(\text{Re } U_1)]^{1/2}}{4}\sum_{k>j\geq 0,\ p\neq 0}\sum_n\int_0^\infty\frac{d\tau_2}{\tau_2^{3/2}}\frac{1}{k}\left(d_1(n)-\frac{d_2(n)}{4\pi\tau_2}\right)\times$$
$$\times\ \text{exp}\left[-2\pi i(\text{Im } T_1)kp + 2\pi i n\left(\frac{j-p(\text{Im } U_1)}{k}\right)\right]$$
$$\times\ \text{exp}\left[-\frac{\pi(\text{Re } T_1)}{\text{Re } U_1}\left(k+\frac{n(\text{Re } U_1)}{k(\text{Re } T_1)}\right)^2\tau_2 - \frac{\pi p^2(\text{Re } T_1)(\text{Re } U_1)}{\tau_2}\right]$$

Then the integral on τ_2 can be carried out with the aid of,

$$\int_0^\infty\frac{dx}{x^{3/2}}e^{-ax-b/x} = \sqrt{\frac{\pi}{b}}e^{-2\sqrt{ab}}\,, \qquad \text{(A.25)}$$

$$\int_0^\infty\frac{dx}{x^{5/2}}e^{-ax-b/x} = \left(\frac{1}{2b}+\sqrt{\frac{a}{b}}\right)\sqrt{\frac{\pi}{b}}e^{-2\sqrt{ab}}\,. \qquad \text{(A.26)}$$

And summing over n, we finally get,

$$\Lambda_{nd} = -\frac{1}{4}\sum_{k>j\geq 0,\ p>0}\frac{e^{-2\pi kpT_1}}{kp}\left(\hat{\mathcal{A}}_f(\mathcal{U})+\frac{1}{\pi kp(T_1+\bar{T}_1)}\hat{\mathcal{A}}_K(\mathcal{U})\right)\ +\ \text{c.c.} \qquad \text{(A.27)}$$

with $\hat{\mathcal{A}}_f$, $\hat{\mathcal{A}}_K$ and $\mathcal{U}$ defined in (2.13), (2.17) and (2.18), respectively.

A.2 The $\text{SO}(q)\times\text{SO}(32-q)$ model

A.2.1 Elliptic genera

The relevant characters for this model are,

$$\chi_o+\chi_v = \frac{E_4^2}{\eta^{16}}\,, \qquad \chi_o-\chi_v = \frac{1}{2\eta^{16}}(\vartheta_3^{q/2}\vartheta_4^{16-q/2}+\vartheta_4^{q/2}\vartheta_3^{16-q/2}) \qquad \text{(A.28)}$$

Then, it is possible to show that,

$$\hat{\mathcal{A}}_{f,1}^{[0,1]} \equiv -\frac{\vartheta_3^2\vartheta_4^2}{4\pi\eta^8}\left(\frac{1}{\tau_2}+\frac{\partial_{\nu_1}^2}{\pi}\right)(\chi_o+\chi_v) = \frac{\vartheta_3^2\vartheta_4^2E_4(\hat{E}_2E_4-E_6)}{12\eta^{24}} = 60-\frac{124}{\pi\tau_2}+\cdots \qquad \text{(A.29)}$$

$$
\begin{aligned}
\hat{\mathcal{A}}_{f,2}^{[0,1]} &\equiv -\frac{\vartheta_3^2\vartheta_4^2}{4\pi\eta^8}\left(\frac{1}{\tau_2}+\frac{\partial_{\nu_1}^2}{\pi}\right)(\chi_o-\chi_v) \\
&= \frac{\vartheta_3^2\vartheta_4^2}{24\eta^{24}}\left[\vartheta_3^{q/2}\vartheta_4^{16-q/2}(\hat{E}_2+\vartheta_2^4-\vartheta_4^4)+\vartheta_4^{q/2}\vartheta_3^{16-q/2}(\hat{E}_2-\vartheta_2^4-\vartheta_3^4)\right] \qquad (A.30)\\
&= 4(q-17)-\frac{q^2-32q-248}{2\pi\tau_2}+\cdots \ .
\end{aligned}
$$

The other modular forms that we need are,

$$
\hat{\mathcal{A}}_{K,1}^{[0,1]} \equiv \frac{1}{4\pi}\left(i\partial_\tau-\frac{1}{\tau_2}\right)\frac{(E_4\vartheta_3\vartheta_4)^2}{\eta^{24}} = \frac{E_4\vartheta_3^2\vartheta_4^2}{24\eta^{24}}[8E_6+E_4(\vartheta_4^4+\vartheta_3^4)+2\hat{E}_2E_4] \qquad (A.31)
$$

$$
\begin{aligned}
\hat{\mathcal{A}}_{K,2}^{[0,1]} &\equiv \frac{1}{4\pi}\left(i\partial_\tau-\frac{1}{\tau_2}\right)\frac{\vartheta_3^2\vartheta_4^2}{2\eta^{24}}(\vartheta_3^{q/2}\vartheta_4^{16-q/2}+\vartheta_4^{q/2}\vartheta_3^{16-q/2}) = \\
&= \frac{\vartheta_3^2\vartheta_4^2}{96\eta^{24}}\left[\vartheta_4^{16-q/2}\vartheta_3^{q/2}\left(8\hat{E}_2+\left(14-\frac{3q}{2}\right)\vartheta_2^4+20\vartheta_3^4\right)+\right.\\
&\qquad\left.+\vartheta_3^{16-q/2}\vartheta_4^{q/2}\left(8\hat{E}_2-\left(14-\frac{3q}{2}\right)\vartheta_2^4+20\vartheta_4^4\right)\right] \ , \qquad (A.32)
\end{aligned}
$$

Finally, we define S and the ST^{-1} transformed forms,

$$
\hat{\mathcal{A}}_{f,i}^{[1,0]}(\tau) = \hat{\mathcal{A}}_{f,i}^{[0,1]}(S\tau) \ , \qquad \hat{\mathcal{A}}_{f,i}^{[1,1]}(\tau) = \hat{\mathcal{A}}_{f,i}^{[0,1]}(ST^{-1}\tau) \, , \qquad (A.33)
$$

$$
\hat{\mathcal{A}}_{K,i}^{[1,0]}(\tau) = \hat{\mathcal{A}}_{K,i}^{[0,1]}(S\tau) \ , \qquad \hat{\mathcal{A}}_{K,i}^{[1,1]}(\tau) = \hat{\mathcal{A}}_{K,i}^{[0,1]}(ST^{-1}\tau) \ . \qquad (A.34)
$$

for $i = 1, 2$.

A.2.2 Degenerate orbits

For the degenerate orbits we have the contribution,

$$
\begin{aligned}
\Lambda_d = -\frac{1}{8}\int_{\mathcal{F}_T}\frac{d^2\tau}{\tau_2^2}\sum_{p,j}\Bigg\{&\left(60-\frac{124}{\pi\tau_2}+\cdots\right)(\mathrm{Re}\ T_1)\mathrm{exp}\left[-\frac{\pi\mathrm{Re}\ T_1}{\tau_2\mathrm{Re}\ U_1}\left|j+\frac{1}{2}+ipU_1\right|^2\right]\\
&+\sum_{r=2,3}\left(4(q-17)-\frac{q^2-32q-248}{2\pi\tau_2}+\cdots\right)(\mathrm{Re}\ T_r)\mathrm{exp}\left[-\frac{\pi\mathrm{Re}\ T_r}{\tau_2\mathrm{Re}\ U_r}\left|j+\frac{1}{2}+ipU_r\right|^2\right]\Bigg\} ,
\end{aligned}
$$

where the dots correspond to order $\mathcal{O}(e^{2\pi i\tau})$ terms not contributing to the final expression. Proceeding as in section A.1.3, we obtain,

$$
\begin{aligned}
\Lambda_d = &-\frac{\pi}{2}\left[5E_{1/2}(iU_1,1)+\frac{q-17}{3}\sum_{r=2,3}E_{1/2}(iU_r,1)\right]\\
&+\frac{\pi}{360}\left[124\frac{E_{1/2}(iU_1,2)}{T_1+\bar{T}_1}+\frac{1}{2}\sum_{r=2,3}(q^2-32q+248)\frac{E_{1/2}(iU_r,2)}{T_r+\bar{T}_r}\right] \ , \qquad (A.35)
\end{aligned}
$$

with $E_{1/2}(U,k)$ the shifted non-holomorphic Eisenstein series defined in (3.21).

A.2.3 Non-degenerate orbits

We begin computing the contribution from non-degenerate orbits of type I. This is given by,

$$\begin{aligned}\Lambda_{nd_I} &= -\frac{1}{4}\int_{\mathbb{C}^+}\frac{d^2\tau}{\tau_2^2}\sum_{p\neq 0,k>j\geq 0}\sum_{n=0}^{\infty}e^{2\pi i\tau n}(-1)^k\\&\quad\times\left[(\text{Re }T_1)\left(d_1(n)-\frac{d_2(n)}{4\pi\tau_2}\right)\exp\left(-2\pi kpT_1-\frac{\pi\text{Re }T_1}{\tau_2\text{Re }U_1}|k\tau-j-\frac{1}{2}-ipU_1|^2\right)\right.\\&\quad\left.+\sum_{i=2,3}(\text{Re }T_i)\left(d_3(n)-\frac{d_4(n)}{4\pi\tau_2}\right)(-1)^{\frac{qk}{4}}\exp\left(-2\pi kpT_i-\frac{\pi\text{Re }T_i}{\tau_2\text{Re }U_i}|k\tau-j-\frac{1}{2}-ipU_i|^2\right)\right]\end{aligned}$$

where we have performed the expansions,

$$\mathcal{A}_{f,1}^{[0,1]}=\sum_n d_1(n)e^{2\pi i\tau n}\,,\qquad\qquad \frac{(E_4\vartheta_3\vartheta_4)^2}{\eta^{24}}=\sum_n d_2(n)e^{2\pi i\tau n}\,,$$

$$\mathcal{A}_{f,2}^{[0,1]}=\sum_n d_3(n)e^{2\pi i\tau n}\,,\qquad \frac{(\vartheta_3\vartheta_4)^2}{2\eta^{24}}(\vartheta_3^{q/2}\vartheta_4^{16-q/2}+\vartheta_4^{q/2}\vartheta_3^{16-q/2})=\sum_n d_4(n)e^{2\pi i\tau n}\,.$$

Proceeding exactly in the same way as in section A.1.4, we obtain,

$$\begin{aligned}\Lambda_{nd_I} &= -\frac{1}{4}\sum_{p>0,k>j\geq 0}\frac{(-1)^k}{pk}\left[e^{-2\pi pkT_1}\left(\hat{\mathcal{A}}_{f,1}^{[0,1]}(\mathcal{U}_1^{[0,1]})+\frac{\hat{\mathcal{A}}_{K,1}^{[0,1]}(\mathcal{U}_1^{[0,1]})}{\pi kp(T_1+\bar{T}_1)}\right)\right.\\&\quad\left.+\sum_{r=2,3}(-1)^{\frac{kq}{4}}e^{-2\pi rkT_r}\left(\hat{\mathcal{A}}_{f,2}^{[0,1]}(\mathcal{U}_r^{[0,1]})+\frac{\hat{\mathcal{A}}_{K,2}^{[0,1]}(\mathcal{U}_r^{[0,1]})}{\pi kp(T_r+\bar{T}_r)}\right)\right]+\text{c.c.}\,,\qquad\text{(A.36)}\end{aligned}$$

with $\mathcal{U}_r^{[h,g]}$ defined in (3.23).

For type II (type III) non-degenerate orbits we proceed in the same way, but performing a change of variables by the corresponding coset representative, $\tau\to S\tau$ ($\tau\to ST^{-1}\tau$), obtaining,

$$\begin{aligned}\Lambda_{nd_{\rm II}} &= -\frac{1}{4}\sum_{p>0,k>j\geq 0}\frac{(-1)^j}{p\left(k+\frac{1}{2}\right)}\left[e^{-2\pi p\left(k+\frac{1}{2}\right)T_1}\left(\hat{\mathcal{A}}_{f,1}^{[1,0]}(\mathcal{U}_1^{[1,0]})+\frac{\hat{\mathcal{A}}_{K,1}^{[1,0]}(\mathcal{U}_1^{[1,0]})}{\pi p\left(k+\frac{1}{2}\right)(T_1+\bar{T}_1)}\right)\right.\\&\quad\left.+\sum_{r=2,3}(-1)^{\frac{jq}{4}}e^{-2\pi r\left(k+\frac{1}{2}\right)T_r}\left(\hat{\mathcal{A}}_{f,2}^{[1,0]}(\mathcal{U}_r^{[1,0]})+\frac{\hat{\mathcal{A}}_{K,2}^{[1,0]}(\mathcal{U}_r^{[1,0]})}{\pi p\left(k+\frac{1}{2}\right)(T_r+\bar{T}_r)}\right)\right]+\text{c.c.}\,,\qquad\text{(A.37)}\end{aligned}$$

and

$$\begin{aligned}\Lambda_{nd_{\rm III}} &= -\frac{1}{4}\sum_{p>0,k>j\geq 0}\frac{(-1)^{k+j}}{p\left(k+\frac{1}{2}\right)}\left[e^{-2\pi p\left(k+\frac{1}{2}\right)T_1}\left(\hat{\mathcal{A}}_{f,1}^{[1,1]}(\mathcal{U}_1^{[1,1]})+\frac{\hat{\mathcal{A}}_{K,1}^{[1,1]}(\mathcal{U}_1^{[1,1]})}{\pi p\left(k+\frac{1}{2}\right)(T_1+\bar{T}_1)}\right)\right.\\&\quad\left.+\sum_{r=2,3}(-1)^{\frac{(k+j)q}{4}}e^{-2\pi r\left(k+\frac{1}{2}\right)T_r}\left(\hat{\mathcal{A}}_{f,2}^{[1,1]}(\mathcal{U}_r^{[1,1]})+\frac{\hat{\mathcal{A}}_{K,2}^{[1,1]}(\mathcal{U}_r^{[1,1]})}{\pi p\left(k+\frac{1}{2}\right)(T_r+\bar{T}_r)}\right)\right]+\text{c.c.}\qquad\text{(A.38)}\end{aligned}$$

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